Elementary Geometry in Non-Elementary Spaces

Counting the Moduli Space of Pentagons on Finite Projective Planes

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Finite Projective Planes

- **Projective planes** are systems of abstract points and lines that satisfy the following axioms:
  I. Every pair of lines is incident with a single point.
  II. Every pair of points is incident with a single line.
  III. There exists some 4 points, with no 3 colinear (a quadrangle).
- The real projective plane satisfies these axioms by letting parallel lines 'meet at infinity' and letting these 'points at infinity' sit on a new 'line at infinity.'
- **Finite projective planes** are structures which satisfy the same axioms as the real plane but have only finitely many points and lines.
- Every finite plane has some **'order'** p such that there are $p^2 + p + 1$ points and lines. In general, if p is prime, π_p refers uniquely to some plane.

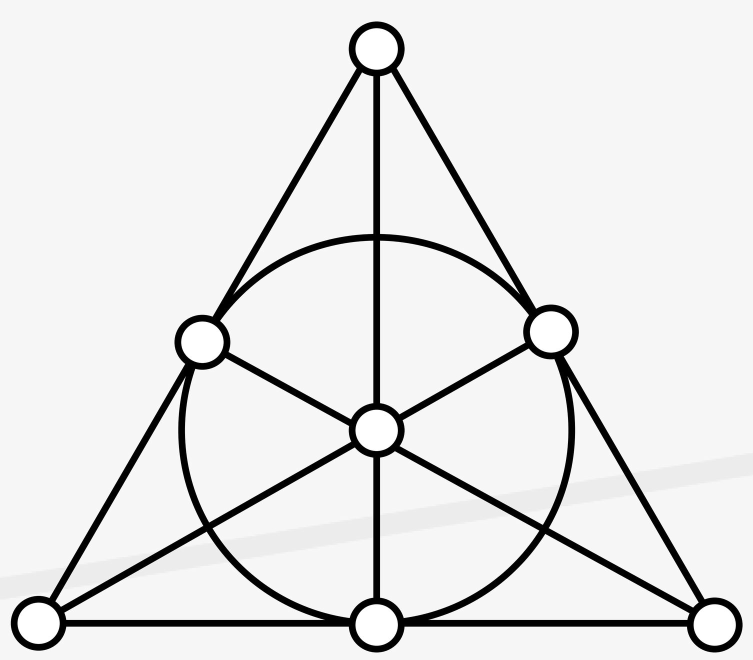


Figure 1: π_2, the Fano Plane

Collineations

- **Collineations** are the natural way of transforming projective planes. They take lines to lines and points to points so that points remain on their respective lines.
- They are fully defined by where they take a quadrangle.
- Alternatively, they are uniquely defined by a 3×3 matrix over the corresponding field, up to scaling.
- If a collineation takes a polygon to itself, we call that a **'symmetry'** of that polygon.
- If a collineation takes a polygon to another polygon, we call those polygons **'similar.'** This is a generalization of similarity in Euclidean geometry.

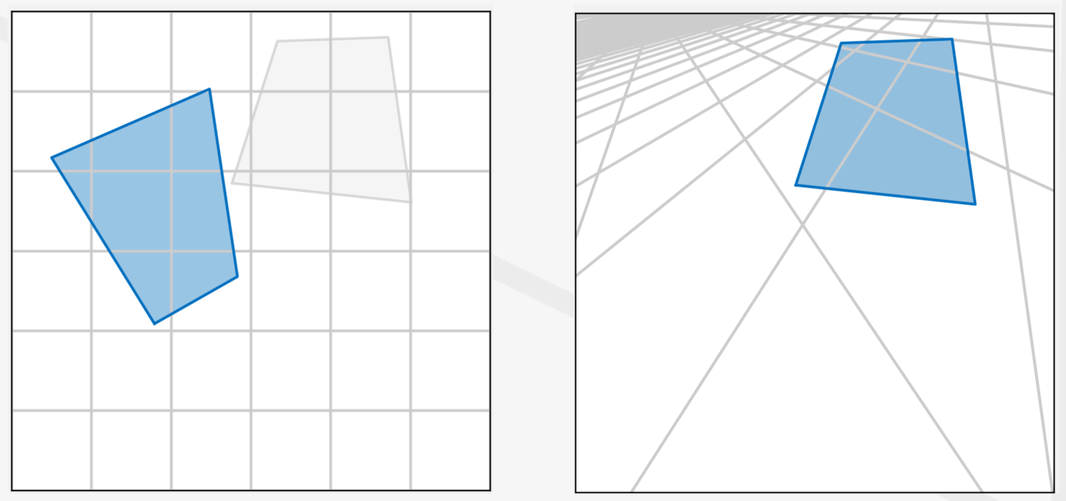


Figure 2: A collineation on the real projective plane defined by where it takes a quadrangle.

Field Planes

- All (known) projective planes can be represented by the subspaces of a 3-dimensional vector space over some field.
- For the real plane, the underlying field is $\mathbb{R}$.
- For finite planes, the underlying field is the finite field of order p, $\mathbb{F}_p$.
- This allows us to apply linear algebra to these spaces, and to give coordinates to a space that doesn't necessarily have a concept of distance.

Moduli Space

- A **similarity class** is an equivalence class of similar polygons. The moduli space is all such classes.
- Similar pentagons have the same number and type of symmetries.
- With collineations, pentagons are the smallest polygons of interest, because it is the first type of n-gon that has more than 1 similarity class.
- Unlike on the real plane, on some finite plane π_p, there are only finitely many of these classes.

Leveraging Symmetry

- We can solve for the 5th vertex of a pentagon if we assume it has some symmetry and fix four of the points. We call these pentagons with given fixed points **'unitary.'**
- This gives us all the unitary pentagons with that kind of symmetry.
- There are $(p-2)(p-3)$ total unitary pentagons on a given π_p.
- Theorem: A similarity class has $\frac{10}{s}$ unitary pentagons, where s is the number of symmetries of that class of pentagons.
- If we choose symmetries such that we only get one pentagon per similarity class, we can leverage this to get the total number of similarity classes.

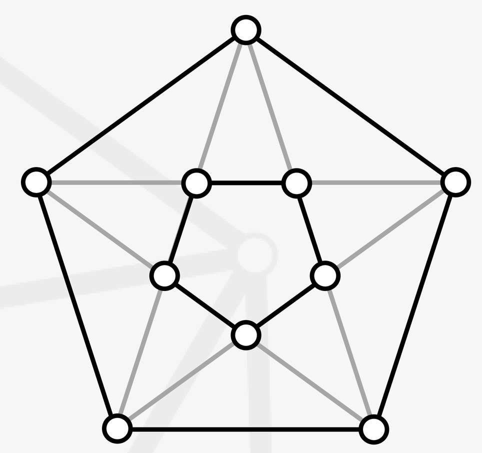
Results

Number of Symmetries	Number of Classes	Reason
10 (Regular)	r_p	Number of 'golden ratios.'
5	0	All pentagons with rotations have reflections as well.
2	$p-3-r_p$	$p-3$ pentagons have reflections, r_p of which are regular.
1	$\frac{1}{10}((p-3)(p-7)+4r_p)$	Remainder of $(p-2)(p-3)$ unitary pentagons.
Total	$\frac{1}{10}((p+3)(p-3)+4r_p)$	Sum of previous.

- r_p is the number of roots of the polynomial $x^2 - x - 1$ on $\mathbb{F}_p$. These roots are the 'golden ratios' of $\mathbb{F}_p$. Excitingly, this means that the relationship between pentagons and the golden ratio transcends metric geometry.
- Using quadratic reciprocity, r_p can be computed directly.

$$r_p = \begin{cases} 1 & p \equiv 0 \mod 5 \\ 2 & p \equiv \pm 1 \mod 5 \\ 0 & p \equiv \pm 2 \mod 5 \end{cases}$$

The Pentagram Map



- The **pentagram map** is a function on the set of n-gons, for some fixed n. It takes a polygon to the intersections of its short diagonals.
- For pentagons on finite planes, it is periodic under repeated application.
- The length of this periodic behavior is unpredictable. However, it is constant across a given similarity class.
- It seems the period of regular pentagons is equal to the smallest n such that $a^n = 1$, where a is a root of $x^2 + 3x + 1$ in $\mathbb{F}_p$, a relative of the golden ratio. I have not been able to prove this conjecture but have checked it for $p < 10{,}000$.

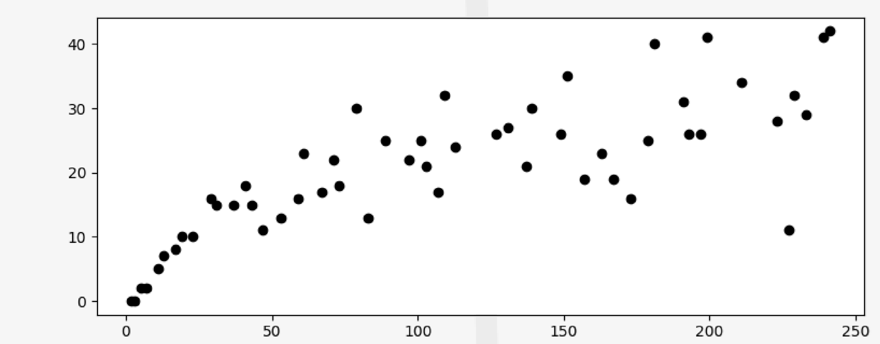


Figure 3: Plane order versus the number of unique periods

Future Work

- I am continuing to try to prove the final conjecture above.
- There is still much to classify about similarity classes on finite planes. So far, I have only been able to differentiate them by the number of symmetries; however, the diversity of pentagram map periods suggests more ways of distinguishing them.

References

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