Tile Invariants and an Exploration of Tilings with Ribbon Pentominoes and L-Pentominoes

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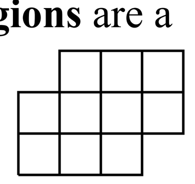
Abstract

In this Independent Study, we survey the mathematics of tiling 2-dimensional regions with polyomino shapes of varying sizes. We investigate tile invariants to prove tileability and examine specific tile invariants, such as the Conway/Lagarias invariant. Using "Tile Invariants for Tackling Tiling Questions" by Dr. Michael Hitchman as a guide for exploration, we survey different techniques for finding tile invariants, such as coloring, boundary words, height, and group theoretic techniques. After this background is established, we answer an open problem posed by Hitchman in the affirmative - we prove the requirements for a modified rectangle to be tileable by area 5 ribbon tiles. In the final part of this project, we consider L-pentominoes and conjecture the requirements for a rectangle to be tileable by this tile set. We prove the conjecture in certain cases.

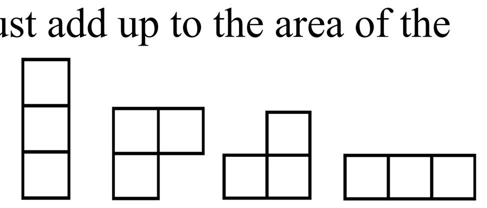
Key Concepts

In this project, we worked with **polyomino tiles,** which are two-dimensional shapes made up of unit squares connected along the squares' edges. A **tile set** is a collection of tiles. One important tile set is the set of ribbon tiles: a **ribbon tile of area n** is a polyomino of n squares laid out in path such that from an initial square, each step either goes up or to the right. We can use a binary string to represent each ribbon tile, where 0 is a step to the right and 1 is a step up.

We place tiles into a **region**, which is made of a lattice grid. A **family of regions** are a collection of regions which share a property. Examples include rectangles and **modified rectangles,** which are rectangles with the top left and bottom right cell removed, as seen to the right.



Given a tile set and a region, we can ask questions about tileability. A region is **tileable by a tile set** if the region can be covered with the tiles from the set without gaps or overlaps. To solve questions of tileability, we use **tile invariants,** which are mathematical properties that remain unchanged no matter how we tile the region. A tile invariant depends on the tile set and family of regions, and it is a linear combination of the different tiles. The **area invariant** states that if all the tiles in the tile set are of area n, then the sum of the instances of all the different tiles must add up to the area of the region divided by n. Another example is the **Conway Lagarias Invariant** for the tile set of ribbon tiles of area 3, as seen to the right. Let us say that a_n is the number of instances of tile t_n. This invariant states that for the family of simply connected regions a_2 - a_3 is a constant number.



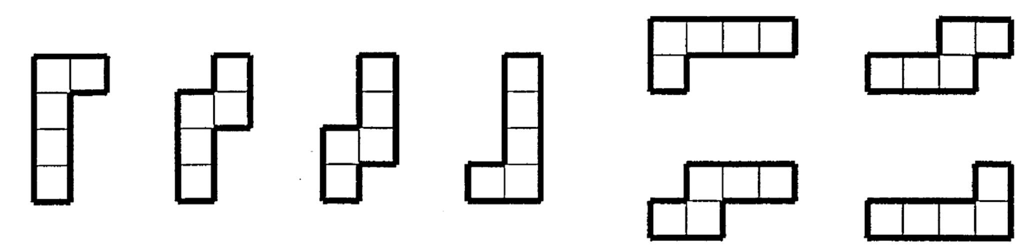
One technique that can be used to find invariants is **coloring.** A coloring is when each cell of a region is given a color, and each color corresponds to some integer or group element. A **color sum** is when we sum up the cells of a specific region or tile. Given an abelian group G with identity element e, a **coloring map** ϕ: $\mathcal{R}_{all} \rightarrow G$ is a function that assigns each region R to its color sum $\phi(R)$. A **T-coloring** is a coloring map ϕ in which $\phi(t) = e$ for each tile t in the tile set T, no matter where t is placed in the lattice. Hitchman proved that if a region R is tileable by T then $\phi(R) = e$.

Another technique utilizes **boundary words.** First, label the edges of the grid lattice, where x is the horizontal edges and y is the vertical edges. These are directed edges. Each path walked through the lattice can be represented as a word, and the path of the outline of the tile is the boundary word. The boundary word of any tileable region R can be expressed as the product of conjugates of the tiles' boundary words. The **winding number** is an invariant of a loop in a plane. In this context, it counts the number of times the boundary path encloses a specific tile.

The **tile counting group** is a group whose elements are tile invariants. We define the tile counting group as $G(T, R) = \mathbb{Z}^n/H$, where H is the normal subgroup of $\mathbb{Z}^n$ generated by all possible difference vectors obtainable from our family of regions R and the tile set T. This group is a useful tool to find invariants because computing the tile counting group is the same as describing all invariants. It can tell us how many total invariants apply to a tile set and family of regions.

Tiling Modified Rectangles with Ribbon Tiles

We considered a question posed by Hitchman: Which modified rectangles M(a,b) can be tiled by the tile set of height-1 area 5 ribbon tiles, as pictured to the right? To start, the notion of **height** is critical for the following results. In a ribbon tile set, each tile is represented by a string of binary, and if we sum that string modulo 2, we have the tile's height. We can use the conditions from the following lemma to disprove tileability based on the notion of height:



The set of height-1 area 5 ribbon tiles has 8 tiles.

- **Lemma 2.7.** *Suppose there exists a tiling of a simply connected region R by the tile set T_n in which an odd number of height-0 tiles are used. Then the set of height-1 tiles of area n does not tile R.*

Pak utilized this lemma to prove a result about the tileability of rectangles with the above tile set:

- **Theorem 3.2.** *The rectangle [a × b] can be tiled by the tile set of height-1 area 5 ribbon tiles if and only if 10 | ab and a,b > 1.*

Hitchman then proved a result tiling modified rectangles with ribbon tiles of area 4:

- **Theorem 3.3.** *Let a,b >1. The modified rectangle M(a,b) can be tiled by the tile set of height-1 area 4 ribbon tiles if and only if a ≡ 2 (mod 4) and b is odd; or a is odd and ab ≡ 2 (mod 8).*

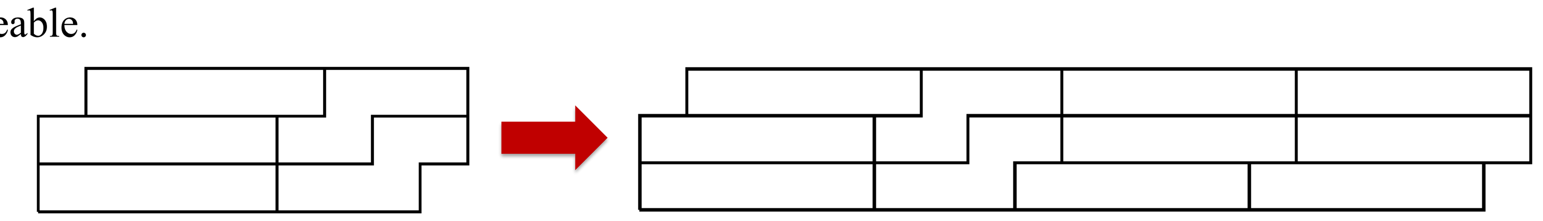
Using the proof techniques used by these two mathematicians, we proved this result concerning tiling modified rectangles with ribbon tiles of area 5:

- **Theorem 3.4.** *Given M(a,b), ab ≡ 2 (mod 10) if and only if M(a,b) is tileable by the tile set of height-1 area 5 ribbon tiles.*

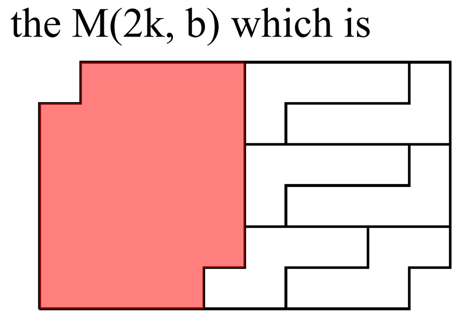
For this proof, we need the following two lemmas. The first follows from the area invariant. The second follows from the symmetries of the chosen tile set.

- **Lemma 3.1.** *If M(a,b) is tileable by our chosen tile set, then ab ≡ 2 (mod 5).*
- **Lemma 3.2.** *Given any M(a,b) and our chosen tile set, if M(a,b) is tileable, then M(b,a) is tileable.*

To prove the forward direction of Theorem 3.4, we must prove $ab \not\equiv 7$ (mod 10). We consider 4 cases, built by the conjugacy classes, and proceed by induction. For example, our first case is when $a \equiv 3$ (mod 10), $b \equiv 9$ (mod 10). We prove M(3 + 10k, 9 + 10l) is not tileable for $k, l \in \mathbb{N}$. In the figure below, you can see that the tiling of M(3,9) has an odd number of height-0 tiles and thus untileable by Lemma 2.7. We can then expand this in either direction to show that M(13,9) and M(3,19) are also untileable.

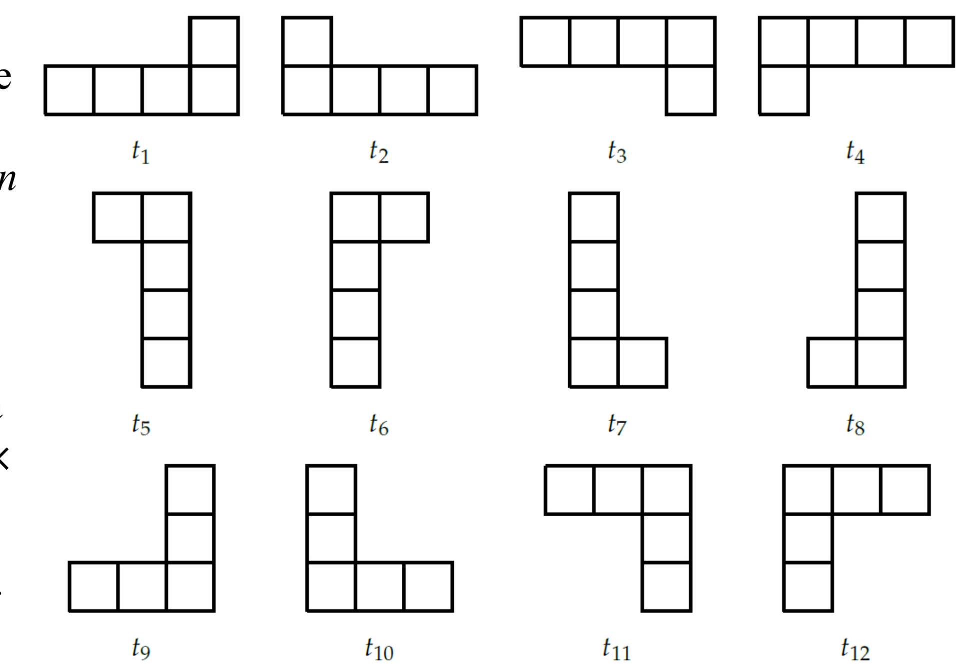


To prove the converse, we also use induction. We assume the height is always even. The base cases are M(2,6), M(4,3), and M(8,4). For the induction step, there are two directions to consider, as we can either expand the height or width. For the first direction, we prove that if M(2k, b) is tileable, then M(2k, b + 5) is tileable. In the figure to the right, the red section represents the M(2k, b) which is tileable. We can expand its width by 5 while retaining the same parity of height-0 tiles used, by adding a column which is the union of M(2,6) and [2 × 5] copies. For the second direction, we prove that if M(2k, b) is tileable, then M(2(k + 5), b) is tileable. In the vertical direction, we can do a very similar expansion but increasing the height by 10 instead of 5.



Tiling Rectangles with L-Pentominoes

We asked what rectangular regions can be tiled by L-pentominoes, the tile set seen to the right. We consider each of these four cases individually: [$2n$ × 10k], [(2n + 1) × 10k], [2n × (10k + 1)], and [(2n + 1) × (10k + 1)].



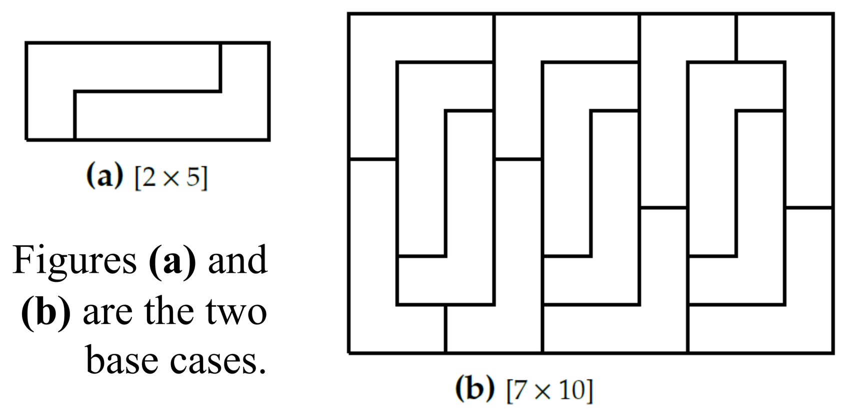
The L-pentomino tile set has 12 tiles.

The smallest tileable rectangles are [2 × 5] and [7 × 10], as seen to the right. It is immediately evident that the two cases with a height of 2n can be tiled by unions of the [2 × 5] rectangle.

- **Theorem 4.1.** *Every [2n × 10k] rectangular region is tileable by the set of L-pentominoes.*
- **Theorem 4.3.** *Every [2n × (10k + 5)] rectangular region is tileable by the set of L-pentominoes.*

For rectangles of the form [(2n + 1) × 10k], we produced the following results:

- **Lemma 4.1.** *Every [3 × 10k] rectangle is untileable by the set of L-pentominoes.*
- **Theorem 4.2.** *Any [(2n + 1) × 10k] rectangular region is tileable by the set of L-pentominoes when n ≥ 2.*



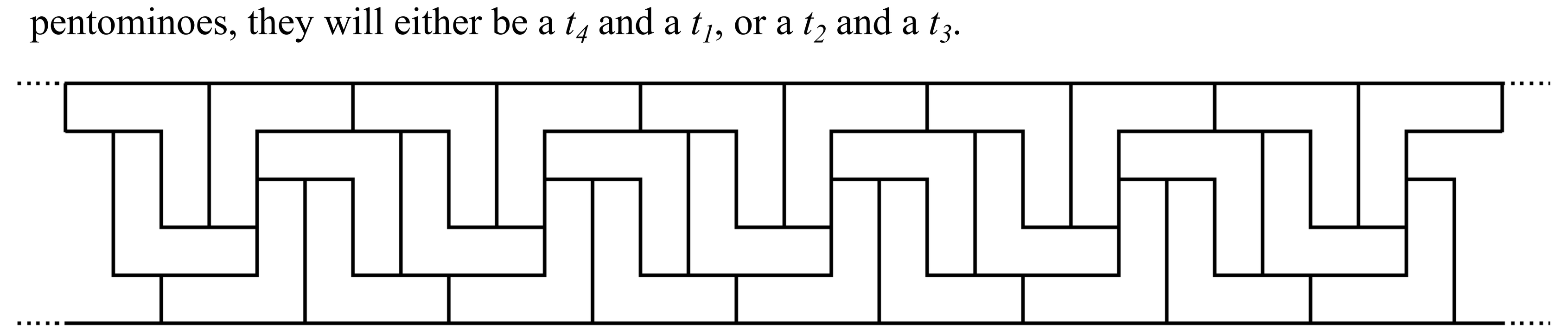
Figures **(a)** and **(b)** are the two base cases.

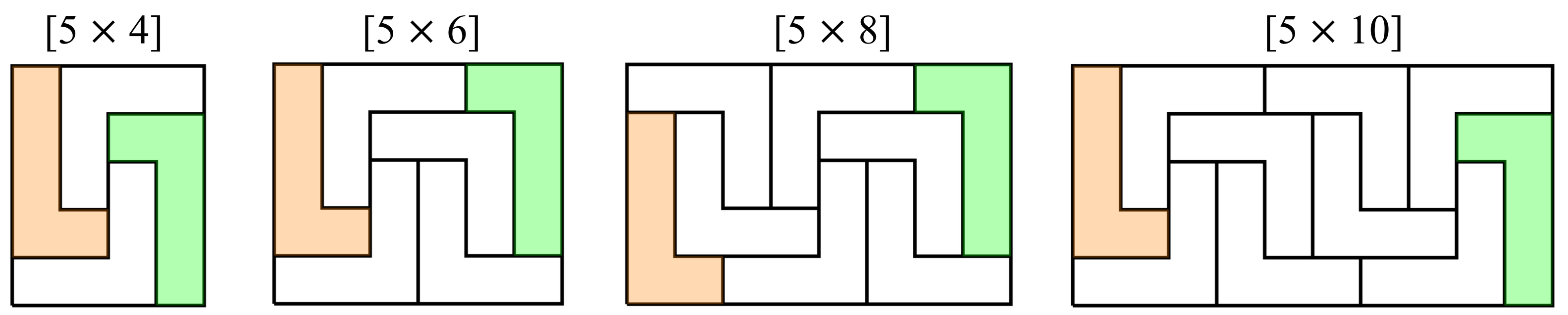
We conjecture that [(2n + 1) × (10k + 5)] rectangles are not tileable. This proof is incomplete. However, we have produced a couple results to this end, concerning the case's smallest rectangles:

- **Lemma 4.2.** *The [3 × 5] rectangle cannot be tiled.*
- **Lemma 4.3.** *The [3 × (10k+5)] rectangle cannot be tiled.*

We also considered the different ways to tile the rectangle, particularly rectangles of the form [5 × 2n]. Using the frieze pattern seen below, we can construct a rectangle of any even width and height 5. We define the Type 1 tiles to be tiles 1-8 and Type 2 tiles to be tiles 9-12. Using this pattern, we proved:

- **Theorem 4.4.** For a [2n × (10k + 5)] rectangular region where n ≥ 2, there exists a tiling with 2 Type 1 L-pentominoes and (2n − 2)(2k + 1) Type 2 L-pentominoes. For the 2 Type 1 L-pentominoes, they will either be a t_4 and a t_1, or a t_2 and a t_3.





Future Work

- The biggest open questions from this research concern the conjecture that are still unproven concerning whether [(2n + 1) × (10k + 5)] rectangles are tileable.
- Consider the set of ribbon tiles of area 6 and the family of modified rectangles. What dimensions must the modified rectangle be for it to be tileable?
- Are there similar frieze patterns to be found when tiling rectangles with L-tiles of area 4 or 6? If so, is there any relationship between that frieze pattern and the one we found for L-pentominoes?

References

- J.H. Conway and J.C. Lagarias. "Tiling with polyominoes and combinatorial group theory". In: Journal of Combinatorial Theory, Series A 53.2 (1990), pp. 183–208.
- Igor Pak. "Ribbon Tile Invariants". In: Transactions of the American Mathematical Society 352.12 (2000), pp. 5525–5561.
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