Anisotropic and Relativistic Diffusion with Partial Differential Equations

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Partial differential equations (PDEs) can describe diffusion systems, where a substance "spreads out" over time. Most of these systems, including classical heat transfer, can be modeled using the traditional diffusion equation. However, some systems require adjusted models that account for anisotropies and relativistic behavior. This independent study examines three differential equations that describe diffusion, including one that combines existing anisotropic and relativistic equations into a new model.

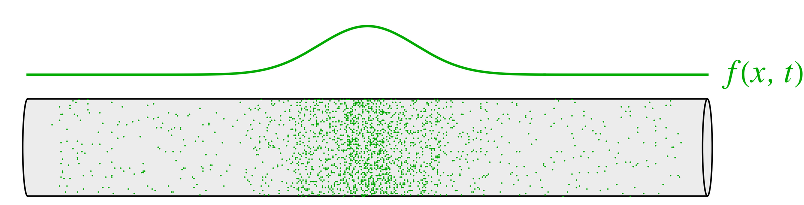
Idea: Combine existing anisotropic and relativistic diffusion models.

Goals:

1. Derive, examine, and solve the differential equations.
2. Demonstrate that the new combined model is relativistic.
3. Solve the PDEs using a variety of techniques.

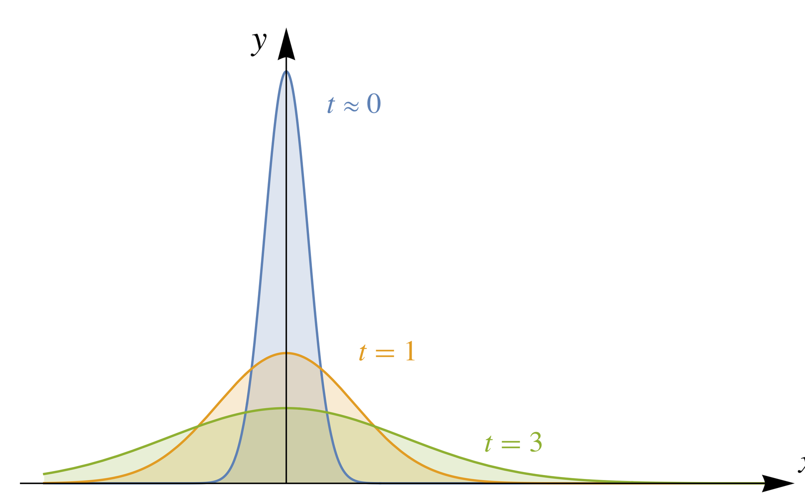
$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}$$

The Diffusion Equation



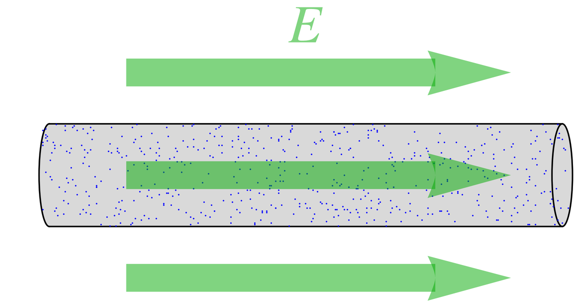
Diffusion describes the way a substance "spreads out"

Heat flow, chemical diffusion, biological processes



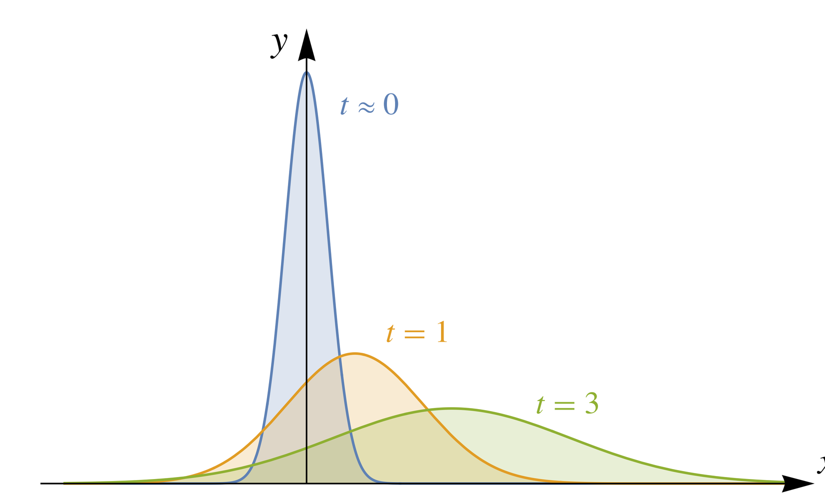
$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2} - \kappa\frac{\partial f}{\partial x}$$

The Anisotropic Diffusion Equation



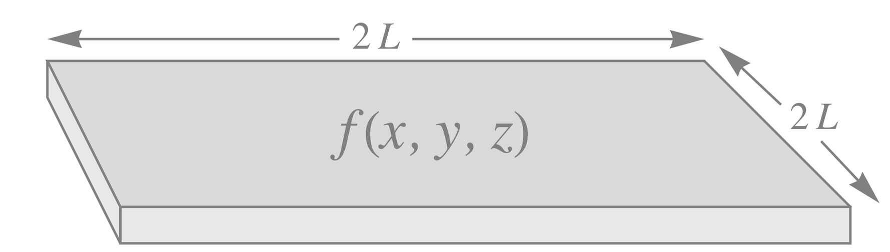
Diffusion with a "drift" anisotropy

Ionized gas in an electric field



$$\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} + \frac{1}{D}\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} - \frac{\kappa}{D}\frac{\partial f}{\partial x}$$

The Relativistic Diffusion Equation



Relativistic AND Anisotropic

Corrects a violation of special relativity

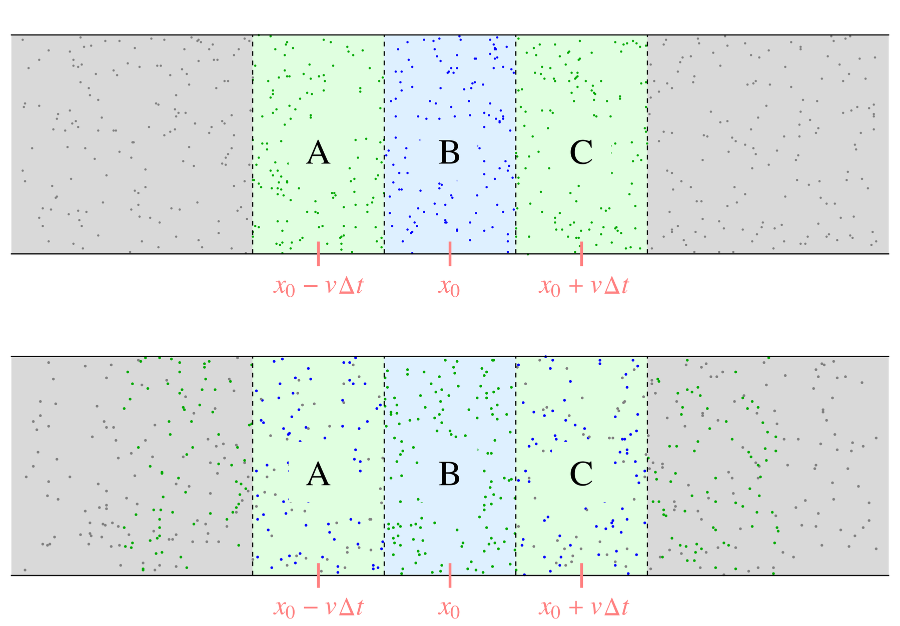
Limits the speed of information travel

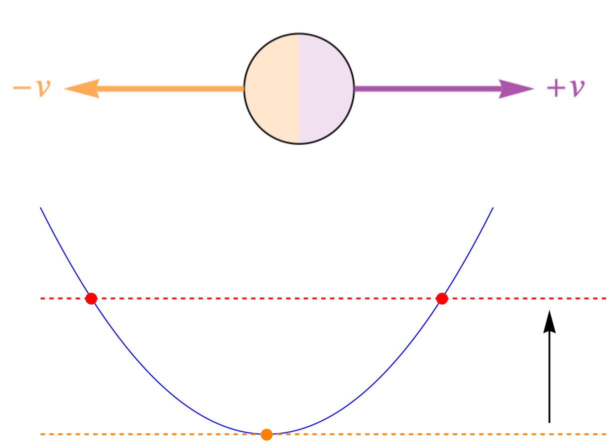
Hyperbolic rather than parabolic

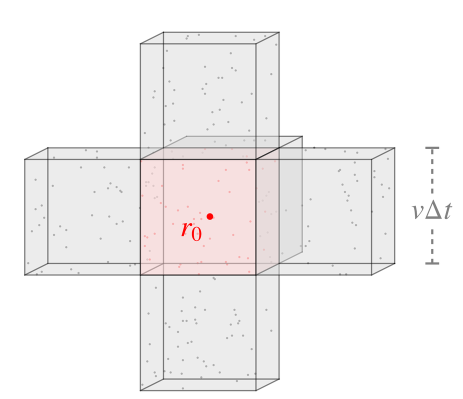
Describes rapid anisotropic diffusion processes

High-energy physics and energy-based ablative techniques

1. Derive Equations







2. Examine Properties

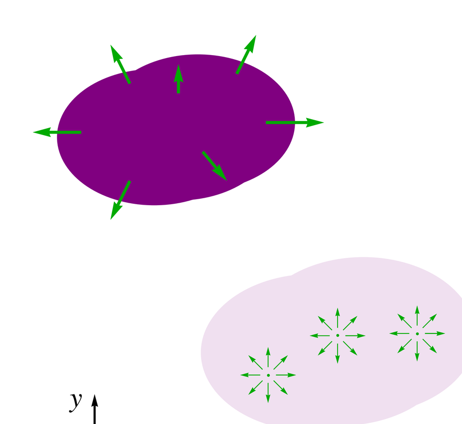
Continuity

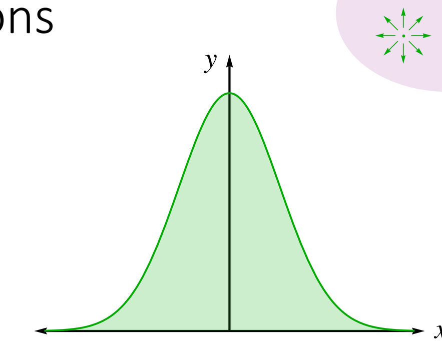
Problems Well-posed

Steady-State Solutions

"Energy Methods"

Vector Calculus





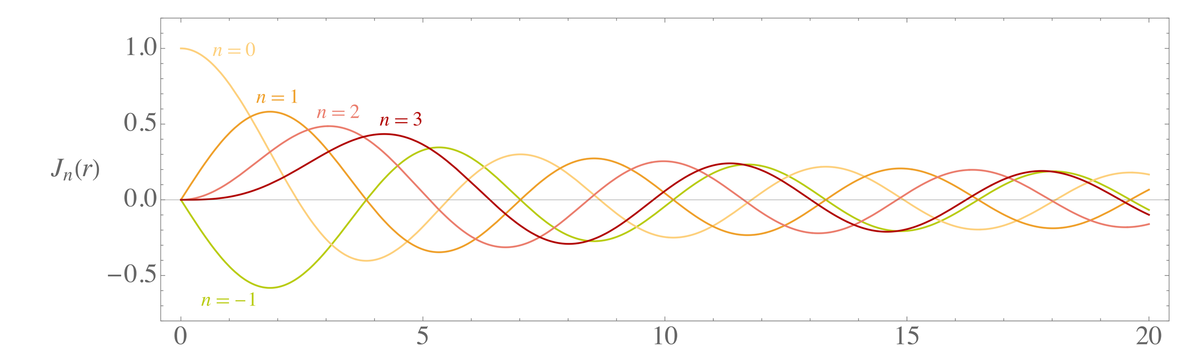
3. Find Solutions

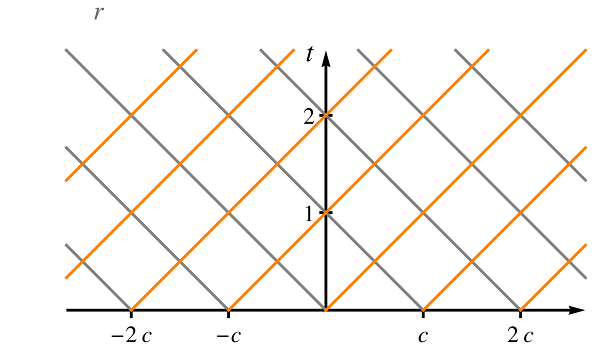
$$f(x,t) = \frac{A}{\sqrt{4\pi Dt}}e^{-x^2/4Dt}$$

$$f(x',t) = \int_{-\infty}^{\infty} g(x^*)P(x^* \to x',t)\,dx^*$$

$$f(x,y,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B_{n,m}\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}y\right)T_{n,m}(t)$$

$$f(x,y,z,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\cos(\omega_x x)\cos(\omega_y y)\cos(\omega_z z)e^{-\omega_t^2 Kt/\rho C}$$





Select Solution Techniques

Separation of Variables	$f(x,y,z,t) = X(x)Y(y)Z(z)T(t)$
Fourier Transform	$F(\omega,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x,t)e^{-i\omega x}\,dx$
Characteristics	$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)f = 0$
Green's Functions	$f(x',t) = \int_{-\infty}^{\infty} g(x^*)P(x^* \to x',t)\,dx^*$

Historical Overview

1822	Fourier	The Heat Equation (Isotropic)
1905	Einstein	Special Relativity
1958	Cattaneo & Vernotte	Hyperbolic Heat Equation (Isotropic)
1944-72	Peshkov, Narayanamurti, Kaminski	Speeds of Second Sound
2005	Ali & Zhang	Relativity Implies HHE
2019	Maillet	Questions HHE Evidence

Main Results

1. New model is both anisotropic and relativistic.
2. New model exhibits "diffusion-like" behavior.
3. Drift anisotropy behaves as expected.
4. Proofs of select properties.
5. Solutions to various PDE problems.

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