Streamlining Speed: A Study of the Mathematics Behind the Aerodynamics of Road Vehicles

Jon Brandt, Advised by Dr. Cynthia Lester
Department of Mathematics
The College of Wooster, Wooster, Ohio

Motivation for Study

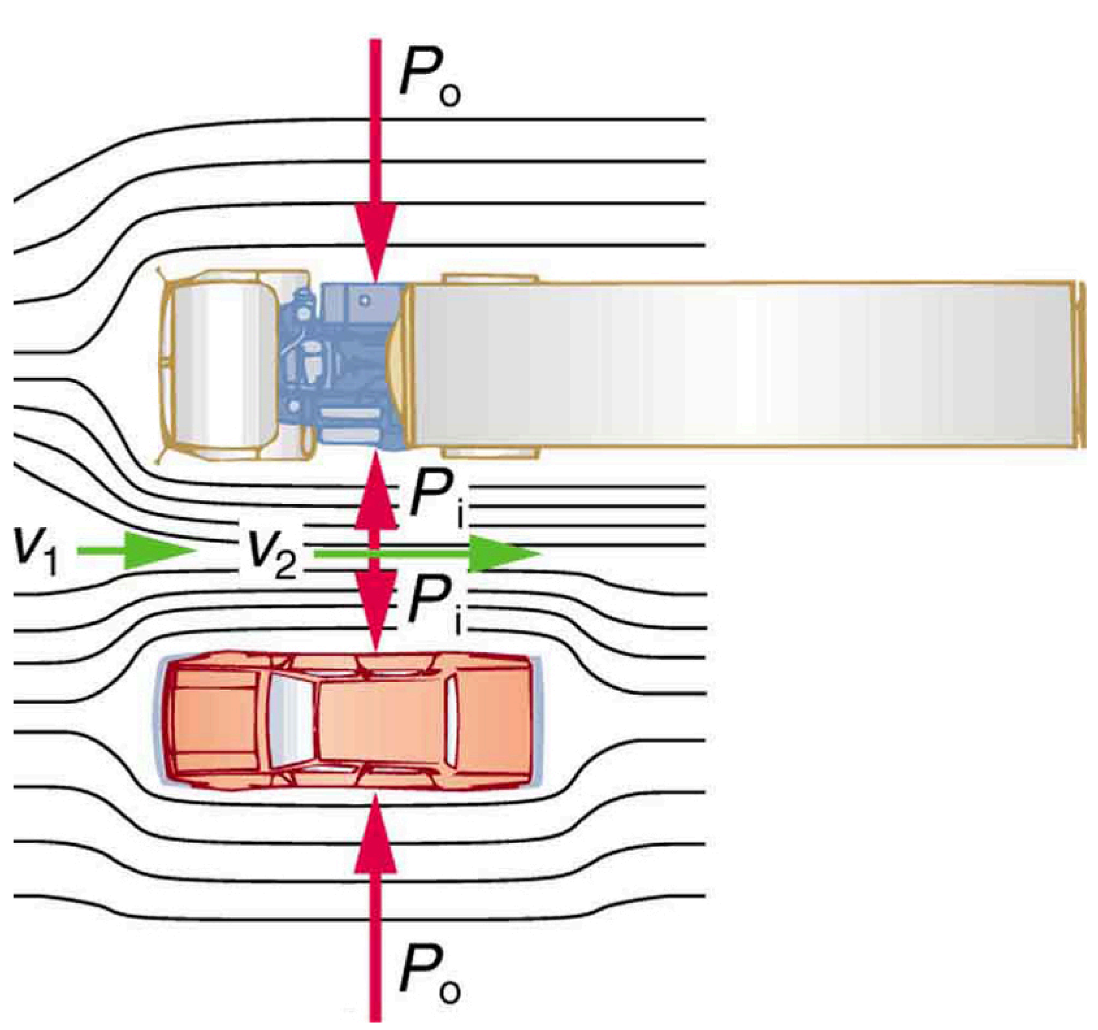
Foundational fluid dynamics equations govern the flow of all fluids and air, making them crucial to our everyday lives, even though we rarely "see" them. These equations lay the groundwork for understanding how air flows around and through objects and how we can use this airflow to benefit vehicle performance and efficiency. Through aerodynamic study, significant benefits can be achieved, from increasing the fuel efficiency of regular cars and semi-trucks to optimizing the acceleration, top speed, and downforce of racing vehicles. Bernoulli's equation is visible in numerous real-world situations, from explaining how airplanes fly and how F1 cars can take corners at extreme speeds to even how a spray bottle works to atomize the liquid. These equations explain the behavior of the air and fluids all around us and can be harnessed to push the boundaries of efficiency and performance.

Introduction

In the field of automotive engineering, aerodynamics plays a crucial role in many important vehicle performance traits such as fuel efficiency, acceleration, top speed, stability, safety, and comfort. This project examines the intertwining nature of math and physics principles that underlie road vehicle aerodynamics and aims to explore the mathematical foundations behind a few significant aerodynamics equations: such as Bernoulli's equation and principle, the Navier-Stokes equations (N-S equations), and the drag coefficient and drag force equations.

Background on Bernoulli's Equation

Bernoulli's equation describes the relationship between the velocity and pressure of a fluid in steady, incompressible flow. Bernoulli's equation also provides valuable insights into how changes in velocity can affect pressure and vice versa around different regions of a vehicle. These insights are especially helpful when studying the aerodynamic effects of lift and downforce, which are governed by Bernoulli's equation. Bernoulli's equation explains how planes produce lift to fly and how race cars utilize massive wings and other features to produce downforce, holding them to the ground. Bernoulli's equation was published in Daniel Bernoulli's book Hydrodynamica in 1738, which later became a foundational text in fluid dynamics, but only after Leonhard Euler published "General Principles of the Motion of Fluids" in 1757 was it possible to solve the problems that Bernoulli had conceptually formulated. [14] Bernoulli's equation can be derived from Euler's equation, Newton's second law, or the conservation laws, which are fundamental principles in physics. [14]



An example of Bernoulli's equation in daily life can be seen when a car passes a semi on the highway. The narrow channel between the vehicles accelerates the airflow $(V_2 > V_1)$, creating a low-pressure region in the vehicles, while the air on the outside of each vehicle has higher pressure $(P_i < P_o)$. This causes the outside air to push the vehicle and semi together, creating a noticeable effect. This same effect has been observed in trains since the 1800s and can also be seen in the bathroom when the curtain sometimes tends to be pulled into the shower. [24]

Derivation of Bernoulli's Equation from Euler's Equation

Euler's equation for ideal (inviscid) fluid flow is: $\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \vec{g}$
where v is the velocity vector, p is the pressure, ρ is the fluid density, and g is the gravitational acceleration vector. To derive Bernoulli's equation, we need to make the following assumptions about the flow conditions:

- Steady flow: $\frac{\partial \vec{v}}{\partial t} = 0.$
- Incompressible flow: This means that the density ρ is constant, and the divergence of the velocity field $\nabla\cdot\vec{v} = 0$ is zero.
- Inviscid flow: No viscous forces (already assumed in Euler's equation).
- Flow along a streamline: We consider the motion of the fluid along a singular streamline.

We'll start with Euler's equation under the above assumptions: [11]

$$(\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \vec{g}.$$

The left-hand side (LHS) of Euler's equation above can be rewritten using the vector identity below: $(\vec{v}\cdot\nabla)\vec{v} = \nabla\left(\frac{1}{2}v^2\right) - \vec{v}\times(\nabla\times\vec{v}).$

Now, if we assume irrotational flow with the condition $(\nabla\times\vec{v} = 0)$, the equation simplifies to: $(\vec{v}\cdot\nabla)\vec{v} = \nabla\left(\frac{1}{2}v^2\right).$

Next, we can substitute the RHS of the above equation into the LHS of Euler's equation at the top to get: $\nabla\left(\frac{1}{2}v^2\right) = -\frac{1}{\rho}\nabla p + \vec{g}.$

We can rewrite the gravitational acceleration g as the gradient of gravitational potential energy, resulting in: $\nabla\left(\frac{1}{2}v^2\right) = -\frac{1}{\rho}\nabla p + -\nabla(gz).$

Now if we move the two terms from the RHS above over to the LHS and factoring out the common term ∇ results in: $\nabla\left(\frac{1}{2}v^2 + \frac{p}{\rho} + gz\right) = 0.$

This equation says that the gradient (how much it expands or shrinks) of the term in the parentheses is zero, meaning that the term itself must remain constant, giving us Bernoulli's equation:

$$\frac{1}{2}v^2 + \frac{p}{\rho} + gz = constant.$$

Bernoulli's equation states that for a steady, ideal, and incompressible fluid, the total energy (the sum of the terms on the LHS) is always constant along a streamline. Think of a streamline as the path that a microscopic fluid particle takes as it travels with a current. Bernoulli's equation only applies on a single streamline and doesn't apply when streamlines cross into each other. The LHS of Bernoulli's equation has three terms, each representing a different energy component. The first term is kinetic energy, the second term is pressure energy, and the last term is potential energy. Thus, the sum of these three energy terms is the total energy of the fluid, and it remains constant along a streamline. [24]

References

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Applications of Bernoulli's Equation

Bernoulli's equation and other fundamental fluid dynamics equations have a wide variety of applications, from passenger vehicles to purpose-built racecars. Bernoulli's equation can be used to:

- Streamline the bodies and individual components of passenger cars to reduce drag and wind noise
- Design air ducts to use the venturi effect, which accelerates the air going through them and reduces pressure, allowing the air to reach critical intake and cooling components faster
- Optimize the shape and angle of aerodynamic features and components on vehicles to produce the right amount of downforce

Increasing downforce comes with an increase in drag, and engineers strive to find the "sweet spot" of the right amount of downforce to increase cornering ability without an excessive amount of drag being produced, slowing the car down.
One of the most significant uses of Bernoulli's equation is to sculpt the undersides and bodies of vehicles to make the airflow under the vehicles faster than the airflow above them. This creates a low-pressure region beneath the vehicle, pulling it towards the ground and increasing downforce. The air flowing above the vehicle will flow slower, creating a high-pressure region that pushes downwards on the body, also creating downforce. Downforce is produced because of the pressure difference between the two air regions above and below the vehicle. [13] This application of Bernoulli's equation is taken to the extreme in open-wheel racing like F1 and IndyCar, where every face and feature of the vehicle is optimized to produce the right amount of downforce. Figure 2 below shows the underside of an F1 car and the various features that produce downforce. These components all work together to produce large amounts of downforce, allowing the vehicle to take corners at high speeds and minimize track times.

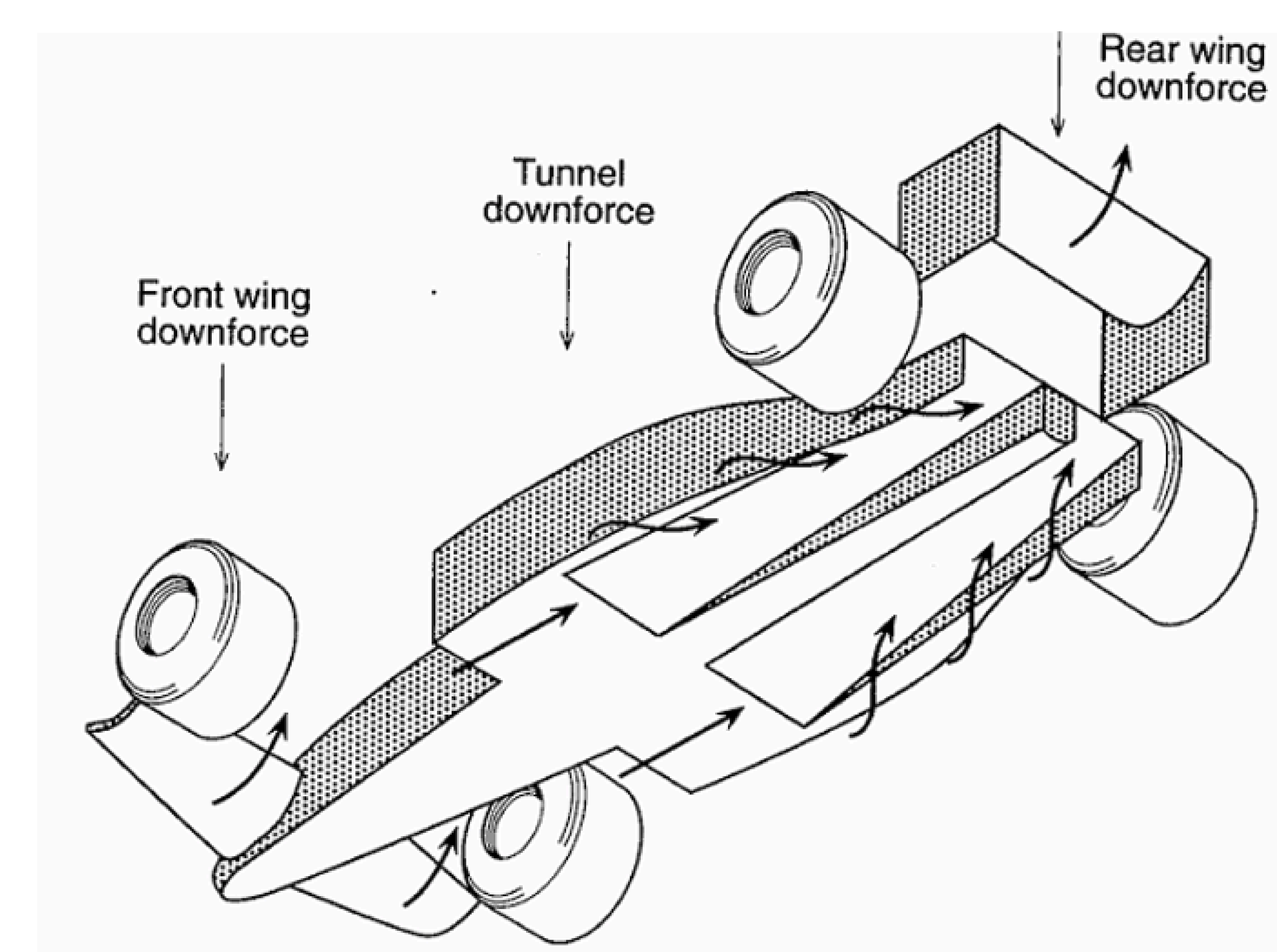

Figure 2: The underbody shaping of an F1 car with front and rear wings and a double tunnel rear diffuser design. [13]

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